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RANDOM NUMBERS FROM A RADIOACTIVE SOURCE

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Random Numbers from a Radioactive Source

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ABSTRACT

The statistical fluctuations in the disintegration of a Cs^{137} source measured by a Radiation Counter Laboratories' 512 multichannel analyzer and a gamma ray scintillator detector were used to produce a sequence of uniformly distributed binary random digits. This binary sequence was then converted to a decimal sequence and seven tests of randomness and uniformity of the distribution were applied.

The results of the tests show that beside statistical sampling fluctuations there is no significant evidence that contradict the hypotheses that:

a) The distribution of the decimal digits fits the uniform distribution (frequency test)

b) There exists no correlation or association among the digits of the sequence (auto-correlation, mean square difference and serial tests)

c) There exists no abnormal clustering or dispersion among the digits (runs tests)

d) There is not a favored five digits arrangement (poker test)

Tables of the uniformly distributed binary and decimal random digits are presented. These random numbers or pseudo-random numbers generated from these may be used in statistical sampling and in Monte Carlo calculations.

The method presented in this thesis to generate a sequence of random numbers compares favorably to other methods that are reported.

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INTRODUCTION

The purpose of this work was to generate random numbers by using the statistical fluctuations in the disintegration of a radioactive source.

A sequence of numbers is called random when there exists no dependence among its members. That is, each number gives null information in guessing other numbers.

The Monte Carlo method consists of solving a physical or mathematical problem by using random numbers to simulate a random process that is directly or indirectly related to the original problem.

Beside their use in the Monte Carlo method, random numbers are widely used in statistics as a safety rule against bias in sampling and also to give validity to tests performed in the design of experiments.

Random numbers are generated by two general ways: arithmetical and physical procedures. Arithmetical procedures consist of iteration methods based upon a mathematical formula that generates pseudo-random numbers. These numbers are only random from the point of view of some specific application or by passing several statistical tests for randomness. Physical procedures consist of a physical device (i.e. dice, a roulette wheel, a radioactive source), used to generate a sequence of random numbers.

REVIEW OF LITERATURE

Arithmetical versus physical processes for generating random numbers

In 1949 J. Von Neumann said "Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin" (1). However, up to now the most common way to generate random number was arithmetical: The congruence method.

Why, if we are convinced that our deterministic mind can not conceive ways to generate random numbers, do we insist on the arithmetical process? The answer in 1959 of the International Business Machine Corporation to this question is that "Fast arithmetical procedures (congruence methods) do exist whose results, though of course not random, never the less do furnish a satisfactory substitute". In addition they are opposed to the generation of random numbers by physical processes because "Nature tends to be systematic, so the construction and maintenance of a mechanical or electronic device - a perfect roulette wheel - is not at all cheap or easy for practical necessities" (2).

A report of Mac Laren and Marsaglia in April 1964 (3) concerned with the suitability of the arithmetical processes of generating random numbers showed that the sequences of pseudo-random numbers generated by arithmetical processes could pass many tests for randomness and still be unsatisfactory when used in Monte Carlo calculations for some practical problems*; they suggest finally that the most suitable sequence of random numbers is that obtained from a table of random numbers, which itself is generated by a physical process.

An answer to the second argument of the IBM Corporation which was the same as that of Brown from RAND Corporation in 1949 (4), concerning the difficulties of generating random numbers by using a roulette wheel is that a roulette wheel is not the only physical process that can be used to generate random numbers, in spite of the fact that RAND Corporation used it for the construction of a 1,000,000 random digits table (5).

Perhaps the strongest argument against generating random numbers by physical processes is, as J. V. Neumann pointed out in 1949, the practical need of reproducibility, that is of repeating the calculations exactly.

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*It seems that there exists physical phenomena where randomness is essential. Nature apparently is not completely deterministic.

If the physical process is incorporated directly into a computer, as it generally is, then this criticism may be answered by noting that it is always possible to print out a given sequence of the computer input random numbers and use this sequence to repeat the calculations. Another possibility is to look for systematic erros in the actual running of the program by performing statistical analyses of the results.

Kahn in his research memorandum of 1954 "Application of Monte Carlo" (6) claims that it is impractical to print a sequence of random numbers generated by a physical process, because of the limited memory and input - output capacity of computer machines. However the practicality of using a table of random numbers (generated by a physical process) for Monte Carlo calculations was demonstrated by Mac Laren and Marsaglia in April 1964 (3). They used different sections of the computer memory alternately and as a buffer, and concluded that a table of random numbers is the most suitable way to generate a sequence of random numbers.

Since it was found that in several cases pseudo-random numbers generated by arithmetical process were unsatisfactory and also, since modern techniques make the use of random numbers generated by a physical process practical; it is now reasonable to start thinking again of better methods of generating random numbers by using physical processes.

Physical processes to generate random numbers

Several investigators who used physical processes to obtain tables of random numbers are; Tippet who used census reports (7), Kendall and Smith who used a mechanical roulette wheel (8), Hamaker who used a rolling 10 sided prism (9), and the RAND Corporation, which produced the largest and most used table by means of an electronic roulette wheel (5).

Research on the use of a radioactive source to generate random numbers was done by; J. Von Hoerner who generated a sequence of random binary digits by considering the position in time of a flip-flop activated by a radioactive source counter (10), and also by M. Isida and H. Ikeda who generated and incorporated into a computer a sequence of decimal random digits obtained from the last digit of a pre-set time radioactive source counter (11).

THEORETICAL CONSIDERATIONS

Experimental observations of a radioactive source show that the radioactive decay occurs at random and at independent moments of time.

It is known that radioactive decay can be represented by a Poisson distribution (12) which tends to the normal distribution with increasing counting rate (13).

In order to use radioactivity to generate random numbers with some specific distribution it is convenient to convert the non-uniformly distributed random numbers obtained by counting a radioactive source, to a sequence of uniformly distributed random numbers. This is accomplished by comparing the successive number of counts of a radioactive source and assigning a one or a zero to the comparison depending upon certain criteria*. By this process the original non-uniformly distributed random numbers are converted into a sequence of uniformly distributed random binary digits, which are then transformed into a sequence of uniformly distributed random decimal digits.

The advantage of this method over the other physical methods mentioned earlier (10, 11) is that the final distri-

*See procedure page 9

bution of random numbers is not affected by the variance of the original distribution**.

If one is interested in incorporating the radioactive process directly with a digital computer, then a slight variation of this method gives faster and therefore better results. For example, the fact that in two successive disintegrations there is the same probability that one takes longer than the other can be used in connection with an electronic clock to feed a sequence of uniformly distributed random binary digits directly to a computer.

**See discussion of results page 26

EXPERIMENTAL PROCEDURE

Generation of the random numbers

A Radiation Counting Laboratories' 512 multi-channel analyzer with a Tally tape perforator and a gamma ray scintillation detector were used to measure the activity of a Cs^{137} source of approximately 1 microcurie*.

The analyzer was used as a preset time scaler in the automatic mode. The information in the channels (counts accumulated during 0.1 seconds) was readout in perforated tape. This tape was converted to IBM cards.

Conversion of the original non-uniformly distributed random numbers to binary and decimal sequences of uniformly distributed random numbers**

A sequence of uniformly distributed binary random digits was formed whose mth term was defined as 0 if N_{2m} was less than N_{2m+1}, or 1 if N_{2m} was greater than N_{2m+1}, where N_{2m} and N_{2m+1} were successive channel counts. Each exclusive set of ten binary digits was then converted to three decimal digits, producing a sequence of uniformly distributed decimal digits.

*See photograph next page.
**The conversion was performed by an IBM 1401 computer.

Figure 1. Photograph of the RCL multi-channel analyzer and complementary equipment used to generate the original random numbers.

Tests for randomness of the decimal digits*

In order to measure the independence of the decimal digits and the uniformity of their distribution, four blocks of 1000 decimal digits each were obtained and tested for randomness using lags 1 through 10. Lag 1 is defined as picking every consecutive number $N_1, N_2, \ldots, N_{1000}$ of the sequence of decimal digits in a block as the random quantity, lag 2 as picking every other number $N_1, N_3, \ldots, N_{999}$ $N_2, N_4 \ldots, N_{1000}$ of the sequence and so on.

Five of the seven tests used are reported in a very recent work of Carlsen (15). They are as follows:

1. The frequency test which consists of calculating chi-square

$$X_1^2 = \frac{1}{100} \sum_{i=0}^{9} (f_i - 100)^2$$

where f_i is the frequency of the decimal digit i (i = 0, 1,..., 9) in the block. The average and variance are also calculated, the obtained values are then compared to the expected values.

2. The serial test which consists of "binning" the numbers into a 10 x 10 matrix. A 1 is added in row i

*The tests were performed by an IBM 1620 computer.

column j, when digit i is followed by digit j (i, j = 0, 1,..., 9). The expected result would be 10 in each position of the matrix. The following X^2 is then calculated.

$$X_2^2 = \frac{1}{10} \sum_{i,\, j=0}^{9} (f_{ij} - 10)^2$$

$X_2^2 - X_1^2$ should be distributed as a chi-square with 90 DF. The expression $Z = (2(X_2^2 - X_1^2))^{1/2} - (2 \times 90 - 1)^{1/2}$ is used as a normal deviate with unit variance and the observed values of Z are then compared to expected values.

3. The auto correlation test which consists of calculating the auto correlation coefficient

$$Ch = \frac{1}{1000} \sum_{m=1}^{1000} N_m N_{m+h}$$

for h = 0, 1, 2,..., 10 where N_m is the mth term in the block and comparing Ch to expected values.

4. The runs test number 1 which consists of finding the runs above and below the mean. To find the runs above and below the mean a sequence of binary digits is formed whose mth term is defined as 0 if N_m is less than 5 or 1 if N_m is greater than 4. A subsequence of k zeros (or ones bracketed by ones (or zeros) at each end forms a run of length k. Runs are counted and compared to the expected values.

5. The mean square difference test which consists of calculating

$$M = \frac{1}{1000} \sum_{m=1}^{1000} (N_m - N_{m+d})^2$$

where d = 1, 2, 3, 4, 5, 10, 100, 101, and comparing M to the expected values.

In addition to these tests, another runs test and a poker test recommended by Brown (16) were used.

6. The runs test number 2 which consists of finding the frequency of runs in the decimal sequence, and comparing this values to expected results.

7. The poker test which consists of scanning the decimal digits in groups of five digits each, simulating poker hands, and finding the frequency of seven classes of hands; busts (symbol abcde), pairs (symbol aabcd), two pairs (symbol aabbc), three digits alike (aaabc), full house (symbol aaabb), four digits alike (symbol aaaab) and all five digits alike (symbol aaaaa). Results are analyzed using a chi-square.

RESULTS

The following results are based on a sample of 4000 decimal digits. Averages are referred to the four blocks of 1000 decimal digits each.

Table 1. Frequency test

Digit		0	1	2	3	4	5	6	7	8	9
Frequency	Observed	432	372	407	381	414	419	375	412	389	399
	Expected	400	400	400	400	400	400	400	400	400	400
Chi-square	Observed	9.165									
	Expected	Less than 16.92 value of X^2 (95 $^{o}/_{o}$) for 9 DF									
Average	Observed	4.48									
	Expected	4.50									
Variance	Observed	.829									
	Expected	.833									

DF = Degrees of freedom

Table 2. Serial test

$\overline{X}^2_2$	$\overline{X}^2_1$	$\overline{X}_2 - \overline{X}_1$	$Z = (2(\overline{X}_2 - \overline{X}_1))^{1/2} - (179)^{1/2}$	Lag
105.65	11.24	94.41	0.363	1
98.30	11.24	87.06	0.184	2
106.80	11.24	95.56	0.446	3
101.50	11.24	90.26	0.057	4
111.20	11.24	99.96	0.760	5
100.54	11.24	89.30	0.714	6
100.50	11.24	89.26	0.017	7
101.05	11.24	90.26	0.057	8
103.75	11.24	92.51	0.223	9
102.10	11.24	90.77	0.095	10

Expected Z (95 °/o) $<$ 1.96

Table 3. Auto-Correlation test

h =	0	1	2	3	4	5	6	7	8	9	10	Lag
Observed $\overline{Ch}$	28.36	19.96	19.97	20.04	20.01	20.00	19.86	19.92	20.00	20.03	20.06	1
	28.36	22.04	20.10	19.91	22.07	20.65	20.01	20.03	19.96	19.98	20.05	2
	28.36	20.08	19.93	20.13	20.07	20.00	20.03	19.97	19.87	19.79	19.94	3
	28.36	20.13	20.11	20.05	20.03	20.14	19.96	19.81	20.09	20.05	19.88	4
	28.36	20.08	20.21	20.06	20.16	20.04	19.99	19.85	19.95	19.98	19.75	5
	28.36	19.94	20.35	20.09	19.95	20.02	20.11	19.91	19.94	19.66	19.97	6
	28.36	20.02	20.08	20.02	19.85	19.89	19.92	19.99	20.07	19.85	19.82	7
	28.36	20.12	20.05	20.00	20.16	19.97	19.99	20.11	20.10	19.71	19.88	8
	28.36	20.15	20.09	19.92	20.16	20.04	19.78	19.93	19.78	19.83	19.74	9
	28.36	20.23	20.19	20.04	20.01	19.82	20.02	19.90	19.94	19.70	19.81	10
Expected Ch	28.50	20.23	20.21	20.19	20.17	20.15	20.13	20.11	20.09	20.07	20.05	

Table 4. Runs test 1

Runs above and below the mean

Length of runs	1	2	3	4	5	6	7	Any	Lag
Observed average	251.75	123.75	75.00	31.50	16.75	7.75	8.00	514.50	1
number of runs	251.75	125.25	67.25	27.50	15.50	7.25	6.00	500.50	2
	252.25	122.00	64.50	32.75	16.25	6.50	6.50	500.75	3
	256.00	120.50	64.75	26.25	14.75	8.75	8.75	499.75	4
	254.00	121.50	61.50	30.50	14.75	9.00	7.75	499.00	5
	250.00	120.50	56.00	33.50	12.75	9.25	8.00	490.00	6
	243.75	127.25	58.00	33.50	12.25	6.50	9.00	490.25	7
	240.00	118.25	68.75	30.25	18.25	5.25	7.75	488.50	8
	250.00	120.25	61.25	30.75	17.00	6.00	9.25	494.50	9
	246.25	118.25	57.25	34.75	16.25	7.25	8.25	488.25	10
Expected number of runs	250.00	125.00	62.50	31.25	15.625	7.80	8.325	500.50	

Table 5. Mean square difference test

d =	1	2	3	4	5	10	100	101	Lag
Observed	16.744	16.671	16.469	16.279	16.057	15.057	15.080	15.164	1
$\overline{M}$	16.693	16.400	16.749	16.383	16.159	16.116	14.616	15.124	2
	16.494	16.768	16.296	16.355	16.430	16.342	15.387	15.074	3
	16.411	16.413	16.472	16.460	16.184	16.336	14.660	14.985	4
	16.508	16.189	16.453	16.227	16.395	16.675	14.868	14.784	5
	16.787	16.400	16.382	16.593	16.403	16.167	11.730	14.901	6
	16.622	16.440	16.507	16.778	16.614	16.501	15.370	15.027	7
	16.434	16.501	16.547	16.205	16.476	17.378	16.557	14.675	8
	16.358	16.422	16.737	16.214	16.392	16.695	14.383	14.959	9
	16.215	16.248	16.546	16.478	16.758	16.517	14.923	14.905	10
Expected M	16.65	16.63	16.62	16.60	16.58	16.50	15.00	14.98	

Table 6. Runs test 2

Length of runs	1	2	3	4	5	Lag
Observed average	821.00	77.50	7.00	0.75	0.00	1
number of runs	826.50	76.25	5.25	1.00	0.25	2
	804.75	80.50	9.50	1.50	0.00	3
	803.75	79.25	10.25	1.00	0.50	4
	800.00	86.00	8.70	0.50	0.00	5
	810.25	79.75	8.50	1.25	0.00	6
	810.25	80.00	9.25	0.25	0.25	7
	823.75	75.75	6.25	1.25	0.25	8
	812.00	77.75	10.25	0.50	0.00	9
	802.75	83.50	8.75	1.00	0.00	10
Expected number of runs	810.00	81.00	8.10	0.81	0.09	

Table 7. Poker test

Class of hand	Bust (abcde)	Pair (aabcd)	Two pairs (aabbc)	Threes (aaabc)	Full house (aaabb)	Fours (aaaab)	Fives (aaaaa)	X^2	Lag
Observed	251	406	81	56	4	2	0	3.272	1
frequency	221	405	94	50	7	3	0	3.264	2
	233	411	75	72	7	2	0	5.463	3
	247	411	64	64	10	3	1	18.545	4
	235	402	79	74	8	2	0	6.663	5
	242	402	78	63	11	4	0	3.377	6
	265	389	78	53	8	4	0	4.380	7
	230	412	86	61	7	4	0	1.032	8
	233	410	95	54	5	3	0	2.297	9
	263	382	85	63	6	6	0	5.281	10
Expected frequency	241.92	403.20	86.40	57.60	7.20	3.60	0.08		

Expected chi-square less than 12.59 value of X^2 (95 °/o) for 6 DF

DISCUSSION OF RESULTS

Frequency test (table 1)

An inspection of table 1 shows that all the digit frequencies are close to the expected value of 400 for a random sample of 4000 uniformly distributed decimal digits. Five of the digit frequencies are above and five below 400. The digit zero has the highest frequency 432, and the digit one has the lowest frequency 372. The even digits have a total frequency of 2017, and the odd digits have a total frequency of 1983.

The chi-square for these digit frequencies has a value of $X_1^2 = 9.165$ which is smaller than 16.92, the critical value of the chi-square for 9 DF at the 95 °/o confidence level. The blocks 1, 2, 3, 4 of 1000 digits each, give the respective values for the chi-squares of $X_{11}^2 = 12.02$, $X_{12}^2 = 10.86$, $X_{13}^2 = 9.44$ and $X_{14}^2 = 12.64$. The average of these chi-squares is $\overline{X}_1^2 = 11.24$. All of the chi-squares have values that are less than 16.92.

The average of the digits is found to be 4.48 which is somewhat lower than the expected value of 4.5 for a uniform distribution in the interval from 0 to 9. The averages of two of the blocks were below and two were above 4.5.

It is interesting to note that the averages reported by Carlsen (15) of 100,000 uniformly distributed pseudo-random numbers obtained by congruence methods and Brown (16) of the 1,000,000 digits of the RAND table (5) obtained by using an electronic roulette wheel are both below the expected mean for the corresponding uniform distribution.

The observed variance was 0.829, a value that is lower but close to the expected value of 0.833. Based on the law of large numbers all of these results should improve with increasing size of the sample. These tests give no evidence to reject the hypothesis that the decimal sequence is uniformly distributed.

Serial test (table 2)

The object of the serial test is to indicate the tendency of given digits to be associated with any other digit. The values of z for the different lags are less than the value 1.96 of z for the 95 % confidence level. This test indicates that the hypothesis of mutual association of digits is rejected.

Auto-correlation test (table 3)

The results from the auto-correlation test indicate that the average values of Ch for different h (h = 0, 1,..., 10) and different lags (lag 1, 2,..., 10) are somewhat lower but nevertheless close to the respective expected values. It seems reasonable to assume that these values should be low due to the fact that the observed average value of the digits is lower than the expected value as is shown in the frequency test.

This shows that there exists no significant evidence of correlation among the digits*.

Runs test 1 (table 4)

Runs above and below the mean give a measure of the tendencies of the digits to cluster or disperse with respect to the mean.

The observation of the values presented in table 4 clearly show that there exists no significant evidence of abnormal clustering or dispersions of the digits with respect to the mean.

*Notice that correlation is not a measure of independence but only of linear dependence (13).

Mean square difference test (table 5)

The mean square difference test provides a measure of the association among the digits.

The average values found of M for the different d (d = 1, 2, 3, 4, 5, 10, 100, 101) and different lags (lag 1, 2,..., 10) are also lower but close to the respective expected values. As with the correlation test the observed values of M should be low since the average of the digits is less than the expected. Therefore there exists no significant evidence of association among the digits.

Runs test 2 (table 6)

The consideration of table 6 gives an indication of the degree of clustering or dispersion of the digits among themselves and shows that this type of anomaly is unlikely to be present in the sequence.

Poker test (table 7)

Table 7 gives an analysis of favored arrangements of five digits, it shows that the values of the frequencies for the different hands (busts, pairs, two pairs,..., fives) and different lags (1, 2,..., 10) except for lag 4 are close to the respective expected values.

Fitness to the expected results is tested with a chi-square, and it is shown that all the values of the chi-squares except that for lag 4 are less than the critical value of 12.59 for the chi-square of 6 DF at the 95 % confidence level.

The anomaly which occurred in lag 4 is easily explained by noticing that this lag has one very unlikely but possible arrangement of fives (aaaaa). The presence of this arrangement gives the main contribution to the large chi-square obtained.

Advantages and disadvantages of the method used in this thesis to generate random numbers

The arithmetical procedures have the advantages that the numbers are easy to generate and easy to reproduce, but have the disadvantage that they cannot generate random numbers.

Pseudo-random numbers generally pass various tests for randomness, but this only proves that we do not know the type of dependence established in their generation, and therefore we are not able to test for it. Nevertheless it is true that in some problems some types of correlation may not affect the results and in those cases arithmetical procedures may be more desirable.

The methods using a radioactive source to generate random numbers have the advantage over other physical methods that the necessary equipment is easy and cheap to maintain and that mechanical devices which gives special tendencies in the sequence of random numbers is eliminated.

As mentioned earlier the method used in this thesis when compared with the two other methods that uses a radio-active source to generate random numbers has the advantage that it does not depend on the variance of the counting rate. M. Isida and H. Ikeda for example who used the last digit of a count to generate a sequence of decimal digits must keep the average counting rate (variance) greater than

$$-\frac{12.4}{\ln(1-a)}$$

in order to have a relative error in the uniformity of the distribution less than a, where a is the maximum difference of frequencies among digits and S. Von Hoerner used the false assumption that the average is in the middle between an odd and even number. The error in the uniformity of the distribution is inversely related to the variance.

Perhaps the strongest assumption made in the methods that use a radioactive source to generate random numbers is that the process of radioactive decay is stationary with respect to time, whereas the activity decreases with time by a factor of $\exp(-\lambda t)$. The factor of decrease $\exp(-\lambda t)$ for the method used in this work (Cs^{137} source, 0.1 second counts) is

$$\exp\left(- \frac{\ln 2}{30 \times 365 \times 24 \times 360} \cdot \frac{2}{10} \right) = 0.9999999999 \doteq 1$$

This shows that the assumption that the process is stationary is a very good approximation.

CONCLUSION

The method used to generate random numbers in this work appears to be better than the other physical method reported in the literature. The physical methods have the advantage over the arithmetical process in that they generate random numbers.

The sequence of uniformly distributed decimal random numbers successfully satisfied all of the seven tests for randomness performed.

Tables of uniformly distributed binary and decimal random numbers are presented in appendix 1 and 2 and are available also as IBM cards that can be used to draw random samples in statistics and these to generate sequences of pseudo-random numbers for Monte Carlo calculations as suggested by Mac Laren and Marsaglia (3). These results suggest to one who wishes to make use of the Monte Carlo method that the construction of a device to generate random numbers using the disintegration of a radioactive source might be incorporated directly into a computer with advantages.

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APPENDIX 1

A table of uniformly distributed binary random numbers

01110011111000001010010000000110100000011110011110001
01010110000110101001101100011000001101100101101011
11111001110011100111001101001111001101001000100010
11100000100111001101100001010010011101010010110011
01100110110000001111110100000001010000110110000100 1
00010100101010011111110110111001111101111000101011
10101101110101110000110000111110000011001000011001 0
00111101010011110100011011001100011000100111101001
11001101100110011110111110010101111110001010110100 1
00111111010010100011011010111001110011101111000101
10110011011000010001001011011001011010000001100001
10001001100100100001111011111100101010010100100011
00000111000100010001001110101111001001111111000010
01101111101000000110101100011101000011110111000011 0
01011001010000100010011001100001001010110010110100
01110110000000100010001011010101011000010000010100 1
01100001100101111101110011000111011010010011110011
00011100101001001101110000100100010100111110101111
00111100011000000000111010101011110111000101011101
01110111010010000100100010001010101111101111001001 00
10001010000000111001001000010111011100101000101000 0
10010000001111110111101110101000100001010010000111 1
01101100110100010001111101011101101000010111111111 0
00010000000110101101011000001000001100010000000111 1
10001000101001000010001011010001111001011010100000
01010001001010110111101010111101010011101010000110
01111011100000100110011101100110011000010111001100
01110010100000001111010101001010111011001010111011 0
11010010011101101110000101001100010100010111111010
10001011101010010111110110011010010100101001010010
01100111010001001001010100111000000001011101111000 0
11010000110010101010110001110101100110000111100111 01
00111111110110100000101001000010000111011011010001 0
01100100000001101100111001101000000001010101011001 00
11010011101011100000001111110101100100011110110111 0
10011111110110111110100011110100101110011010001011 1
10111001000111000110000001011110110000110001110011 0
00111101000101100110100000001101101000110001101110 1
01010101010010111010000001100110010000110000100110 0
01010101100110010101000101001011101101000111001011
11001010101100000111001000110010000110111110100111 0
11111010101011111001010011100000001011000110111101 0
01001011101010000011001100000010011111010101001101 01
10000101011101100010100111000111001110100000000001 1
11001100001101100010001101010110100100101111111110

A table of uniformly distributed binary random numbers (Cont.)

```
10110001001111001101111110110000110110010100010010
1101100110100110111010010101100110111010000101111111
111111101010000101011100010110111000000000110001010
1000101010101011111101110100100011101111110110010101
0000000000100010100011110000110101011001011111101101
01011110001000110111011101111110101100110000001011
0101010001000000111100000001111010000000100001101000
0010001110011101100000001000101011111111110001100000
1000000000111110011100011100100011111110010010110011
0110011110011001100110111100100101001010011100 1111
0111001100111010101001000010100100101000011110 0101
0100100011100101110100110111000001010011000001 1000
01001101010011011011100100101100001010000001 1110100
00110000000111001111111111101011111010111010 1101 1100
1110011100010010011011111111011111100101110011 11111
11010101010100101011100011110011101000101011111000
11000110000001111111111100110010110010110110011 1001
00001010010111100100011010101010000110010111 1000001
10110001011010101010100001110010100011111111 10111
00100011101111100111010000000101001111101101010000
1000100000100100111011101001001010111011101001 0000
1100101011001010101111000101001000100111011000100
0001011001010100101100111111100001000101000000 1000
0101111100011110000010000000010100000000011 0000000
10001101111000001111011010111110100110000000000 10111
00010010000100010100000011001010011011110101 000001
0011100101101101010011010110001000111010110100 1110
100101011100111100001001000011011001011110101011011
0000101101111011001011010100110101001111001010111 00
0011101011000110110000111100110011110011111100001 11
11010011011101101110000110111101111111001100110010
1111000100111110010000100111111101000111100000 01001
00101001111100110111011100011101000110111110011 1111
1001010000100110110000110100011110010101101001 0011
0011110111011000100101000100100111000100101101 0111
10101110111001110111101111110000100100110011010100
010110110000001110111100110101011111011100011 11010
1111010010111110001001010100101110011100000000 1111
1010101000010001101110100010011001110010111001 0101
0010010111110011010100110010010001111100001111 101
0100110011110001000111100101000001000011000100 1000
01000011100110010001110100100001100101011101110010
000000101110011100111100111011111111000110111 00001
110101101010010101100001001000111010001011011010 1000
10010111100101110101101100110010100110011011001111
```

A table of uniformly distributed binary random numbers (Cont.)

```
1100000000101101110000110000111111001010010011101001
01011110101110011101001000100001101110101100010110
001011011001011110101101011010010001110000111101000
01011010000100110110100100111000100110000110111000
11001111011100100100000110000010111011000001110001
01101111110010111110010100101000111101001000101111
10010010100011111101001010010110011000111001010010
100001000010001010001000101010001111001101001001100
10100011010110010110110100010111010101001001100111
0111001101111000101000101100110001010111100011101
11010100111100001010111110101001101111001111000000
000110011011111011001100000011000110000000001001001
11110010100100101010011010011001001101100100011011
0110101001001100000010000101111011001000000111110 10
0011010010111111001111110010010011100000110111101000
01110001000001110110100100100110011110100101111110
00111011010001110010011000001010101010111111000011001
10101001001011000101111101110000100110001100010000
010111001010011011111000001010011010110101111110010
01111111101000010001101100010001100010101111010010
101111010111111001001000000111100101001011110101010
011110111010010000011110011110111010100110011001110
01111101000100110010001011111100001001001101111010
010000001111111111100001011101111101001101100011111
011111001110111100101111001000111100100000011101000
11011111110100010010010010001111000101100100110100
011100110000010000001000000100001100111001111001011
00011011011001001001110000101011110000100101100011
111101000111011110101000000100000000100111111100011
101100011100101111010000010010001000000000101010111
111001110110011010011010011101100111011000110010100
00101000110001111110111001101001010100111011110001
11111011101100010101100101010011101111101101011100
1101000110110000000111000101000001110111111001000010
011100111100001011101011001101110100110111110110110
10101101001110101001011010110000110011110011011111
11111011101100000000011010010011101010000001101100
11010111010101001011111001100010101110110011101100
100110101100000100101110101110101101111111011000101
110001000000011111101111100001100010101000001101101
011111001100000010000000101111110000010100110011111
01010111000110111001100100011111010011100001000000
001000000010110100011000101011011111110011001101101
0010000111000100111111011111000100111110000001001100
0010110111111010111000111000110111001101100111111011
```

A table of uniformly distributed binary random numbers (Cont.)

01111110110111000100101011001101001010000101110 11
01011000011111100110001111100111111110001011011101
0110100101100110110001100000111110011010100000100
11101001001011100101100100111010000011010111000111
00000011100110100011100100010111101100011100101100
00011111111001001011000000000011000111111101010110
11100110001011011101001100101000110011001000111101
1011110100101111011100100111010010111011001001011
01100010110100001101100101010000100011110000010001
11111101011111100001111010110111010110011100111011
00001100000011010000100111000111010001111010110101
11010111110111101011000111111010011000010001001100
11001111100011111001000000110000111111110100010001
01100111011010011100111100010000000110011011010 0110
10000000010111110010101100111001100101111110100 1010
110110100110010010010110100001101100101011110 10110
11000000101111110101100110011111000111011100011 0111
01011111110100011000011011010011110000110100000111
11011000111000010100101011110000010010110001110011
00011101001000110111100110010000110010000001001001
00000100111011000100110010001100000001101010101011
11110010011100001100100110011001010010000011000100
011111100110111110010001000101100000000001011010000
00110001101011001100110001001000010101101100001110
11110111001111001011111001110101110111011001000010
011101001110010111101110110111101001111000110010110
100000001101111101111010010110100101011111101000001
01000110010101100101001111110011000000111100101011
11000010110101100011011001000001100010010010100000
11011111110010101100000100101100001111100011100110
01000010000110011100000100001110000100101110101000
0010110001000101110000010110100101010011100010011001
00110010111110011001101110011010110111001010000001
00001000110000100011100110000010001111111101010011
101000000110111111011011000101110011101000110011100
111100010001011100001111001001110010111111110011111
000100000110010010010110111100111111111001100110001
0111100101101110011100101000111111111011001110010000
001001011011111001110101111100100110001010011110011
011100011010100000000000101001101011001011111110101
11010101000100111010010011100101000110100110010110
01111011011100000111111100100011111001110101000101
01100011000111110110100010110100011010001000101001
10111000111101010000110000011110111001010010100100
00110110000100100101010101110011010111011100100010

A table of uniformly distributed binary random numbers (Cont.)

10011011100010110110111011110101001011001000001001
10100001101110111010001010100110011010000110001111
11011100111010110011111101010000001011101001000111
10101100001111110100111100101111001111110001110011
10110001100001010000000111011110010011101000111111
10110010101100110100001000101001101011100000000110
00101111011011010001001110110101000000000000101101
01001101101001011100101101101000000000001110011001
0100000001001010010001000011110010000000001001100100
00001111100111000001010011111001101110011010100001
00000000011010001000110100110000100011110010001111
10011010110111111110000000110010011111110100100010011
00001011000100011000111010000001111001011011111010
00010100011011011010010000100100010001000111010010
110001001100110010111100001101001001000111111101101
01110010000001010011100111000001010011111010101101
1000011010000001100101101011010100111111010101101
00110011111010100010010000010110000010111111011101
1100001010110011000001111101101011001010111111110
011010101111101111111111100001110011101011010100100
01011101001001000001101101010101100111111111000101
101000000000111100100010101111011101101001101100101
01110001111000000010110011010000111011011000011010
01111001111101110001110111111001010110011111101000
10101101100100001000110001101000111011110011001000
101111111000110011011010000101111111101110000011100
100010011110100110011010000001010000000001110011111
100110000010001101000101101110010101010000101101011
0001000000111110011011110101011000100111101111101
0100010010110110101100011110000100100011000110000
1110011111000000001001100101001110111001001011001001
0100111100100011010010011011011001101001110000011
111101100111000100111001010111111010001110011001111
01110011111000011101111001000110010011000100001011
00110110101101100101100011010110001100001000001011
011101110011010000010001101011100101100111110111010
00101001110101001100111100001111010011101011011010
0100100100000101100111101011000011000100100011110
00111111110001100000010000000011000100111101111110
11011110110000000110010010100000010100011001000010
10010100001101010001011011010001011001000000011100
01011101000100010111010101010011011101110101111101
0010000111000110010101010101001010011011001010111101
11111100101001110110111000011000101110010111001011
111000110000000100000001101110001111101100101111010

A table of uniformly distributed binary random numbers (Cont.)

```
01011011101011110011100011101101011000110001100110
01101111101001111011111011010100111110100111100111
00111011011011101010110000011100110010100000110100
00110100001111111001010010110111010110000001010100
10000100101110010010000000110011011001010111011001
100001111101110010000001110000100101101111011110001
00000110100111001100101100010001000011011101110101
11011010011101100001101111110010001001101100011100
00000011101111101001110101100000110011100000001001
00000001110110101010001110000110100010001111000010
00010000011010100010100011100111011110110100110 1
001010000110101010000010011111111010101001110110001
00101010000001010111111001110111110001010110100100
10101000000010001100000111101010011100111101000111 0
000010000101110011111000110000101000110110000001010
001001101001111111110110111110101111010001111010010
00010011011000111100100101001011101000100111100010
00100011010011000011001111001000101011011100000100
00011101011010011011101000001011101111010011010110
01010000011110110011101001111011001011101011101101
01000101010001010100110101100101110001111101000011
11010011010010101100100101110100110101111001011 0000
00001010000010010110000011000110000100001100001000
11101101011110000101001100100101101010100100111011
01110101101010001000111000100101111001111111100110
1111111010101010111111111110010111111110100010100
10111000010001011101010011101100000001110010001001
10101100010110101110100000100011010001101100000100
00111010000110111010011000101001010111101011 00010
00010010101000101101111011101111011011111000111001
00001011010000010100101010110100010111011010001111
11111100011000001111010101001100101000001100100001
11000001101010000001010100111111111011100001111 01101
00100000100110100011100110101011100010100000011 1011
00101100101111001011011111011000110100100000011 01011
11000000100001001101110010011000110010011101111100
01110011000011110111101001011000001000011110101111
00101111101110001001111110111000001100001110011101
011011110010110011010000010111110000001011001111111
00110111000111001000110010011101001101000100001001
01100011100101100001001000101000111101001011110001
00010101110001011010011010100101000010001011001100
01110001001001001111110111000111101101101000010110
11001101101001010001111101011101010101110110011111
001110100101110000000100101011101110011101101001100
```

A table of uniformly distributed binary random numbers (Cont.)

11110000001101101001011000000011111011010110110011
01101011111011011110011000010110000101010011001001
1101001011111001010011010010001111110001101011011
111001011101000101011000101000100101101111111101011
111010000001000111111001000000100101111101011010011
110001001010101100110110111101111100010101011100110
10010101000000101110011001100111010000100110010100
101110100011001100011101000001110000010011111101111
110010000110110111100101101001110100101000010110 11
01011100100111010101000000000110000111101001000 0000
01111000100010111100110111001000010101100101100101
0001001100011000101101000100110111111100001 10111001
11111001011011000011010101110101101101010101101 0010
000101000010010110001000100010110010010001110110110
110111001111011000100010001101011100100110010000110
110001001011011011010011110010111000001011001110 10
0110010000011111010110001110110110001100100100 00000
100101010100110011101010111111110111001111000100101
001111111010111110010010101000011011100111001110 0001
01111000101100001101110100010011011001001110111 11
01111000000000010001101000010011110101111000111100
001110100000100011101100101010000010010100010 0110
10110111001101011000001011110110100001101100011111
00010100111101001001100100101001010011110101101000
0100111100111010110001010010010100000001111111101111
001110011000010011111100100111110111111110011010 0111
011110111011011100000100110101100101010111000110100
100011001011111100101000101010100111011001110000000
01010000000100010110100011011001101100010110001011
00000101100011011000110111000110010011011101111011
01001110010110010000111101101100110011111010101101
01001111000010010111010001010011111000011011110111
00011100100000110010001101101101110011001101111100
01101001000100000011001110110100100001101101 0001001
111001010001001011111111000010100100000011010011 0011
010010001001110100110011000001010101111111010000010
00001011000001110011111111100111010100011001111 00
11011111100101110101110010001111100001110100110 0001
11001101110100100100010111001001110111100110011110
11111101110111110110111011101010111001100110001011
110101001110101100001101110100110111100101011 10000
10100011100110000010100100001111001011000010101001
101101101010101101000010100100000000101111010000100
10001110101101110000011001001111000011001011010101
10010101010010011001110101011111111100110110100 0110

A table of uniformly distributed binary random numbers (Cont.)

110111101111000110100011010001000110110100111111000
001001111010100110100110001111100101101001100110000
1101111000000101111001110010000000100110111011001
11000000011010001110100110010101000110111010001010
0000010010111110010000101011100111001111100100000
1010000010010110011001001010011101010101101111101
111011101010111010100011001101101010110001001110010
10111111001010000001101011111010110111100011000101
011111110000001011001011101110111110111110101100011 0
010000001011100011101111110011100101110101000100100
10111111111101001010101110111010111100111010000011
1011111110001110101010000001001100000011100000111100
00111011000000100110110000101110010100111011100110
01010111110100100101101001001000100111000000110111
1101001000000011100000000100101000011110000001010101
01011010101110100000000000111111101010001100010100010
01111011100000011111100111100100111111101101111010000
11111100101001011010110010000000110101011101010111
11000110111111111110001000011101011111111010010000001
011111111110000001110110010010011101111110100110010
00111010011101101011111001111111011001000111111101110
10100100001111100100110100011101110011001000111001
00110111010011100000111101111001100010101000100011
01011101010000111001010100100000011000010011011000
000001100010111100011001011001010001100100011101000
11010001010000011011000111110101111100101111000100
0101101001100101100101001001010011000110011000

APPENDIX 2

A table of uniformly distributed decimal random numbers

46302051852724134442571005436399923121182054690092
30406291794110316406467770826718785035556953687835
37050245244435098489822414997497361253163430462965
28995916929102827323591596244652539854290235703440
82991804720683627060413903818178732431145897770839
43241512938088349477132546763804552114267741080577
98341751006442938609801554657818048567232469568733
46464940384736094604580306774733748418780830815065
58663870594594413073334011752835171117609925255417
02135684673650680387863944657037127974045404739023
02443585188382213411860511340763424050829484598107
82268539846761312088890301268770501309533709227826
70897321727487062117287238353379089639455470293247
94050005529633454933765674796910113370150152581041
42472069511096513974228505179414409754330463460938
29160522008302430921958704024419292794273292429496
58318532995729307607921279214068250414568530509617
09677270323142999257318848544590932699656811171778
33125406900838048051264038456703086330402307227605
06233212297248565708465992251348153470458696786703
48235108243243903845878111508818962484159655009167
20991765924739327445272769963189214721236405982150
31229789943389240156802836496269171518212005741253
07785916067072270401840405882011627827995449858600
66571976944039180923337892513644279018237369256848
83603105901524408298040967481134471903302445595862
53165611594528593084486588653406837852615461906177
96010277109607397029842231028342519304740025021091
44514884521185856343822371143866875376767099881527
84370734040958510529708394978757996527165938494579
50030619103689025909393479949975496896823289527429
17903084600650084129711095857789623559778905220099
95711186036756343925617667236404163126922339753789
77155223957846304671784595069293742820722376842093
61088613319706992366190189427357097842539630841094
99008095522415348441583846064129419086621135159760
45492325150745268966018735397249744542162038792251
69327415905254550144195819458121275870007998549207
33202204573756759154374299355269596143017993941857
83969386349112660907683024902425527341366896405142
25134987184078428735854177149827702075715673832804
36963263867532700075115116567612196073019708803006
19650576106951272019871678608678298897192547757846
76069513164065154953011758332813572527718117793554
00393160895172075062230264412067530936177184361334

A table of uniformly distributed decimal random numbers (Cont.)

```
03503560857585164176339562841296436896981592706714
68898174857183279121509747611972434546400833578523
14313282406493775968999325396502557104553739848775
29334886180762218295730052164695416961639988369198
00465836889718311157100802391571277148201384300061
89721237256045310604730000921258328572512612062449
84414314361950805050627504428092848576208173026506
84667872035382914560836243356855380508586362222076
74261523989778816502690766427895963629676372577725
94886945551482023753848795289517269426479423920076
44112670285516656412578316091047326803630659739815
51765274875120145560926009595740713316794310820899
20746354191358826721886956556053447683421517995416
73329638467302920899400982862541925135517668910122
96081578592849436356028372279340887381135101340667
90801622050237836675557135510244663594925048723774
67752020522083018560845309140257149535434562251837
53026460708769885873865764550796014856206009007426
67457047884516168015985294516808792596542067228024
52316540334635606535221545108934889780775725949304
82141195242173772117667642957214321947670814749277
60243117604015004952877694990120142631547064890548
79986877672767373026695191376894305208387721162211
97175354074557955879569045020685093655527339160801
95249313041961890605917897149242010777007780620189
24602476620339431909050489254447170475232552204568
07308265398353138244753087090425264716452591881950
41523344859946284496087338443543173438953320184391
68408599192775526039282875771907237866914459067425
96092793834404744817833772801734361842091370469003
48218882086357076395275504441997707346874722208068
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